Formal Metatheory of Second-Order Abstract Syntax

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Lemma (Weakening) If $\Gamma \vdash t : \beta$, then $\Gamma, x : \alpha \vdash t : \beta$.
Proof Trivial.

Lemma (Substitution) If $\Gamma, x : \alpha \vdash t : \beta$ and $\Gamma \vdash s : \alpha$
then $\Gamma \vdash [s/x]t : \beta$.
Proof By induction on $\Gamma, x : \alpha \vdash t : \beta$, weakening, and exchange.

Theorem (Type preservation) If $\Gamma \vdash t : \alpha$ and $t \rightsquigarrow s$ then $\Gamma \vdash s : \alpha$.
Proof By induction on $t \rightsquigarrow s$ and the substitution lemma.

Lemma (Renaming) If $\Gamma \vdash t : \alpha$ and $\rho : \Gamma \rightsquigarrow \Delta$, then $\Delta \vdash \langle\rho\rangle t : \alpha$.
Proof By induction on $\Gamma \vdash t : \alpha$.

Lemma (Weakening) If $\Gamma \vdash t : \beta$, then $\Gamma, x : \alpha \vdash t : \beta$.
Proof By renaming with $\Gamma \rightsquigarrow (\Gamma, \alpha)$.

Lemma (Lifting) If $\Delta \vdash \sigma : \Gamma$, then $(\Delta, \alpha) \vdash \text{lift}\, \sigma : (\Gamma, \alpha)$.
Proof By induction on $\Delta \vdash \sigma : \Gamma$ and weakening.

Lemma (Simultaneous substitution) If $\Gamma \vdash t : \alpha$ and $\Delta \vdash \sigma : \Gamma$,
then $\Delta \vdash [\sigma]t : \alpha$.
Proof By induction on $\Gamma \vdash t : \alpha$ and lifting.

Lemma (Substitution) If $\Gamma, x : \alpha \vdash t : \beta$ and $\Gamma \vdash s : \alpha$
then $\Gamma \vdash [s/x]t : \beta$.
Proof By simultaneous substitution with $\Gamma \vdash \text{id}, t : \Gamma, \alpha$.

Theorem (Type preservation) If $\Gamma \vdash t : \alpha$ and $t \rightsquigarrow s$ then $\Gamma \vdash s : \alpha$.
Proof By induction on $t \rightsquigarrow s$ and the substitution lemma.

Lemma (Lift-id) $\mathsf{lift}_v\ \mathsf{id}\ v = v$

Lemma (Ren-id) $\langle \mathsf{id} \rangle t = t$

Lemma (Lift-var) $\mathsf{lift}\ \mathsf{var}\ v = \mathsf{var}\ v$

Theorem (Identity substitution) $[\mathsf{var}]t = t$

Lemma (Lift-ren-ren) $\mathsf{lift}_v\ (\varrho \circ \rho)\ v = \langle \mathsf{lift}_v\, \varrho \rangle (\mathsf{lift}_v\, \rho\, v)$

Lemma (Ren-ren) $\langle \varrho \rangle (\langle \rho \rangle t) = \langle \varrho \circ \rho \rangle t$

Lemma (Lift-ren-sub) $\mathsf{lift}\ (\langle \rho \rangle \circ \sigma)\ v = \langle \mathsf{lift}_v\, \rho \rangle (\mathsf{lift}\, \sigma\, v)$

Lemma (Ren-sub) $\langle \rho \rangle ([\sigma] t) = [\langle \rho \rangle \circ \sigma] t$

Lemma (Lift-sub-ren) $\mathsf{lift}\ (\sigma \circ \rho)\ x = \mathsf{lift}\ \sigma\ (\mathsf{lift}_v\ \rho)\ v$

Lemma (Sub-ren) $[\sigma](\langle \rho \rangle t) = [\sigma \circ \rho] t$

Lemma (Lift-sub-sub) $\mathsf{lift}\ ([\varsigma] \circ \sigma)\ x = [\mathsf{lift}\ \varsigma](\mathsf{lift}\ \sigma\ x)$

Theorem (Substitution associativity) $[\varsigma]([\sigma]\ t) = [[\varsigma] \circ \sigma]\ t$

Proof assistants can demand an intimidating amount of rigour

Proof assistants can demand an intimidating amount of ~~rigour~~ boilerplate benefit from

Proof assistants can ~~demand an~~

~~intimidating~~ amount of ~~rigour~~

a tasteful ~~boilerplate~~

code generation

Syntax description file

```
syntax Λ

type
  N   : 0-ary
  _≻_ : 2-ary

term
  app : (α ≻ β)  α   →   β
  lam : α.β          →   α ≻ β
```

$\Longrightarrow$

Data type of types and terms

$$
\begin{aligned}
&\mathsf{data}\ \Lambda\mathsf{T} : \mathsf{Set}\ \mathsf{where}\\
&\quad \mathsf{N} \quad : \Lambda\mathsf{T}\\
&\quad \_\succ\_ : \Lambda\mathsf{T} \to \Lambda\mathsf{T} \to \Lambda\mathsf{T}
\end{aligned}
$$

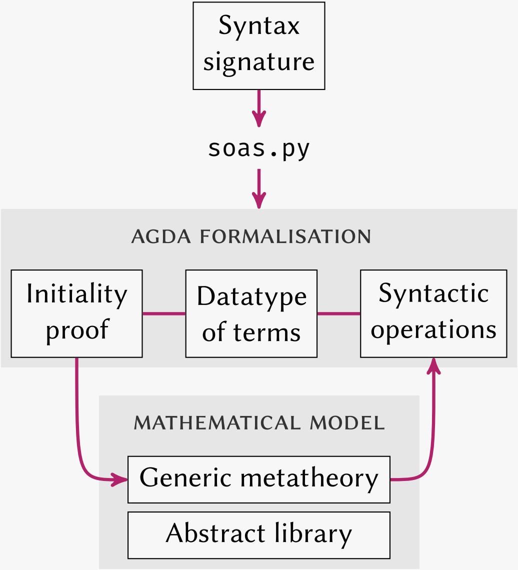
$$
\begin{aligned}
&\mathsf{data}\ \Lambda : \Lambda\mathsf{T} \to \mathsf{Ctx}\ \Lambda\mathsf{T} \to \mathsf{Set}\ \mathsf{where}\\
&\quad \mathsf{var} \ : \mathcal{I}\,\alpha\,\Gamma \to \Lambda\,\alpha\,\Gamma\\
&\quad \mathsf{app} : \Lambda\,(\alpha \succ \beta)\,\Gamma \to \Lambda\,\alpha\,\Gamma \to \Lambda\,\beta\,\Gamma\\
&\quad \mathsf{lam} : \Lambda\,\beta\,(\alpha \cdot \Gamma) \to \Lambda\,(\alpha \succ \beta)\,\Gamma
\end{aligned}
$$

Syntactic and semantic operations

$$
\begin{aligned}
&\mathsf{wk} \quad : \Lambda\,\alpha\,\Gamma \to \Lambda\,\alpha\,(\beta \cdot \Gamma)\\
&[\_/] : \Lambda\,\alpha\,\Gamma \to \Lambda\,\beta\,(\alpha \cdot \Gamma) \to \Lambda\,\beta\,\Gamma\\
&[\![\_]\!] \quad : \Lambda\,\alpha\,\Gamma \to \mathcal{M}\,\alpha\,\Gamma
\end{aligned}
$$

Correctness laws

$$
\begin{aligned}
&\mathsf{syn\text{-}sub\text{-}lemma} \ : [r/]\ ([s/]\ t) \equiv \big[\ [r/]\ s\ /\big]\ ([r/]\ t)\\
&\mathsf{sem\text{-}sub\text{-}lemma} : [\![\ [s/]\ t\ ]\!] \equiv \mathcal{M}.sub\ [\![s]\!]\ [\![t]\!]
\end{aligned}
$$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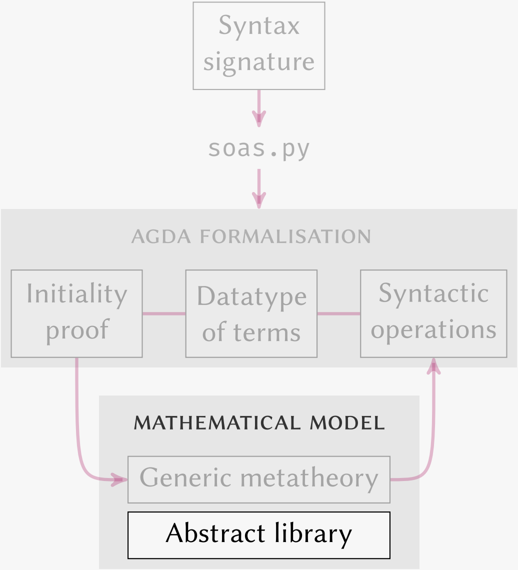
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The universe of discourse of abstract syntax is typed- and scoped- sets – *sorted families*

Objects and morphisms

$$\mathsf{Family}_s : \mathsf{Set}_1$$
$$\mathsf{Family}_s = T \to \mathsf{Ctx} \to \mathsf{Set}$$

$$\_\!\Rightarrow\!\_ : \mathsf{Family}_s \to \mathsf{Family}_s \to \mathsf{Set}$$
$$\mathcal{X} \Rightarrow \mathcal{Y} = \{\alpha : T\}\{\Gamma : \mathsf{Ctx}\} \to \mathcal{X}\;\alpha\;\Gamma \to \mathcal{Y}\;\alpha\;\Gamma$$

Example (Variables)

$$\begin{array}{l} \mathsf{data}\ \mathcal{I} : \mathsf{Family}_s\ \mathsf{where} \\ \quad \mathsf{new} : \mathcal{I}\;\alpha\;(\alpha \cdot \Gamma) \\ \quad \mathsf{old}\ \ : \mathcal{I}\;\beta\;\Gamma \to \mathcal{I}\;\beta\;(\alpha \cdot \Gamma) \end{array}$$

Example (Syntactic terms)

$$\begin{array}{l} \mathsf{data}\ \Lambda : \mathsf{Family}_s\ \mathsf{where} \\ \quad \mathsf{var}\ \ : \mathcal{I}\;\alpha\;\Gamma \to \Lambda\;\alpha\;\Gamma \\ \quad \mathsf{app} : \Lambda\;(\alpha \succ \beta)\;\Gamma \to \Lambda\;\alpha\;\Gamma \to \Lambda\;\beta\;\Gamma \\ \quad \mathsf{lam} : \Lambda\;\beta\;(\alpha \cdot \Gamma) \to \Lambda\;(\alpha \succ \beta)\;\Gamma \end{array}$$

Context maps assign terms of a family to variables in a type-preserving way

$$
\begin{array}{ccccccc}
\Gamma = & \alpha & \beta & \gamma & \delta & : \mathsf{Ctx} \\
 & \big\downarrow & \big\downarrow & \big\downarrow & \big\downarrow & \\
\sigma = & t_1 : \mathcal{X}\,\alpha\,\Delta & t_2 : \mathcal{X}\,\beta\,\Delta & t_3 : \mathcal{X}\,\gamma\,\Delta & t_4 : \mathcal{X}\,\delta\,\Delta & : \Gamma \; {-}[\mathcal{X}]{\rightarrow} \; \Delta
\end{array}
$$

Simultaneous substitutions represented as a function space

$$
\begin{aligned}
&\_{-}[\_]{\rightarrow}\_ : \mathsf{Ctx} \longrightarrow \mathsf{Family}_s \longrightarrow \mathsf{Ctx} \longrightarrow \mathsf{Set} \\
&\Gamma \; {-}[\; \mathcal{X} \;]{\rightarrow} \; \Delta = \{\alpha : T\} \longrightarrow \mathcal{I}\;\alpha\;\Gamma \longrightarrow \mathcal{X}\;\alpha\;\Delta
\end{aligned}
$$

Renamings are variable-valued context maps

Contexts and renamings form a cocartesian category

$$
\begin{aligned}
&\_\rightsquigarrow\_ : \mathsf{Ctx} \longrightarrow \mathsf{Ctx} \longrightarrow \mathsf{Set} \\
&\Gamma \rightsquigarrow \Delta = \Gamma \; {-}[\; \mathcal{I} \;]{\rightarrow} \; \Delta
\end{aligned}
$$

Families can be parametrised by context maps

Make $\mathcal{Y}$ dependent on a $\mathcal{X}$-valued context map into an arbitrary Δ

Internal hom of $\mathcal{X}$ and $\mathcal{Y}$ in the skew-closed category of families

$$\langle\!\langle \_,\_ \rangle\!\rangle : \mathsf{Family}_s \longrightarrow \mathsf{Family}_s \longrightarrow \mathsf{Family}_s$$
$$\langle\!\langle \mathcal{X} , \mathcal{Y} \rangle\!\rangle \; \alpha \; \Gamma = \{\Delta : \mathsf{Ctx}\} \longrightarrow (\Gamma \; \text{-}[\; \mathcal{X} \;]\!\rightarrow \Delta) \longrightarrow \mathcal{Y} \; \alpha \; \Delta$$

"Renamable" terms are elements of $\Box\,\mathcal{X} \triangleq \langle\!\langle \mathcal{I}, \mathcal{X} \rangle\!\rangle$

A term $t \in (\Box\,\mathcal{X})\;\alpha\;\Gamma$ applied to $\rho : \Gamma \rightsquigarrow \Delta$ gives $t\;\rho : \mathcal{X}\;\alpha\;\Delta$

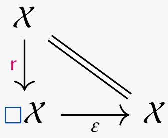
"Substitutable" terms are elements of $\langle\!\langle \mathcal{X} , \mathcal{X} \rangle\!\rangle$

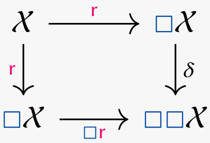
A term $t \in \langle\!\langle \mathcal{X}, \mathcal{X} \rangle\!\rangle\;\alpha\;\Gamma$ applied to $\sigma : \Gamma \; \text{-}[\mathcal{X}]\!\rightarrow \Delta$ gives $t\;\sigma : \mathcal{X}\;\alpha\;\Delta$

Renamability is coalgebra structure

Definition

A *coalgebra* for the comonad $\square$ on $\mathbf{Fam}_{\mathrm{S}}$ is an object $\mathcal{X}$ and a structure map $\mathsf{r} : \mathcal{X} \rightarrowtriangle \square\mathcal{X}$ compatible with the comonad structure:





$t \mapsto \mathsf{r}\ t\ \mathsf{id} = t$

$t \mapsto \rho, \varrho \mapsto \mathsf{r}\ (\mathsf{r}\ t\ \rho)\ \varrho = \mathsf{r}\ t\ (\varrho \circ \rho)$

The structure map $\mathsf{r} : \mathcal{X} \rightarrowtriangle \square\mathcal{X}$ acts as the *renaming operation*

It turns a family with coalgebra structure into a renamable family

$$\mathsf{r} : \mathcal{X} \rightarrowtriangle \square\mathcal{X} = \forall\{\alpha\,\Gamma\} \rightarrow \mathcal{X}\,\alpha\,\Gamma \rightarrow (\forall\{\Delta\} \rightarrow (\Gamma \rightsquigarrow \Delta) \rightarrow \mathcal{X}\,\alpha\,\Delta)$$

Renamability is coalgebra structure

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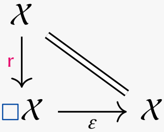

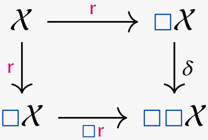

$t \mapsto \mathsf{r}\ t\ \mathsf{id} = t$

$t \mapsto \rho, \varrho \mapsto \mathsf{r}\ (\mathsf{r}\ t\ \rho)\ \varrho = \mathsf{r}\ t\ (\varrho \circ \rho)$

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$$\mathsf{r} : \mathcal{X} \rightarrowtriangle \square\mathcal{X} = \forall\{\alpha\,\Gamma\,\Delta\} \to \mathcal{X}\,\alpha\,\Gamma \to (\Gamma \rightsquigarrow \Delta) \to \mathcal{X}\,\alpha\,\Delta$$

Substitutability is monoid structure

Definition

A *monoid* in a closed category $(\mathbf{Fam}_{\mathrm{S}}, \mathcal{I}, ⦇-,=⦈)$ is an object $\mathcal{M}$ with a unit $\eta : \mathcal{I} \rightarrow \mathcal{M}$ and multiplication $\mu : \mathcal{M} \rightarrow ⦇\mathcal{M}, \mathcal{M}⦈$ satisfying unit and associativity laws.

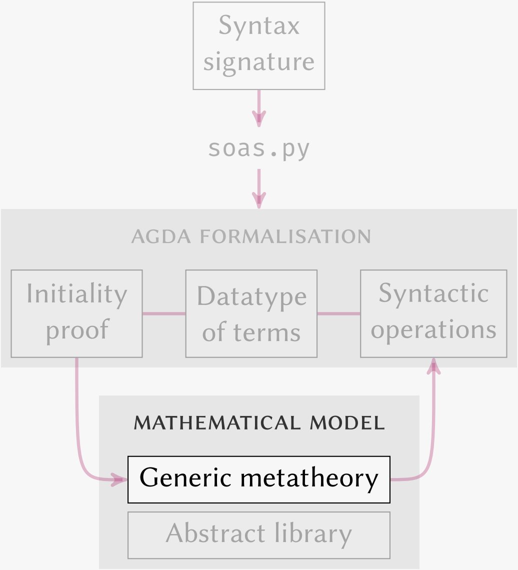
```
record Mon (𝓜 : Family_S) : Set where
  field  η : 𝓘  ⇾ 𝓜
         μ : 𝓜 ⇾ ⦇ 𝓜 , 𝓜 ⦈

         lunit  : {σ : Γ –[ 𝓜 ]→ Δ} {v : 𝓘 α Γ} → μ (η v) σ ≡ σ v
         runit  : {t : 𝓜 α Γ} → μ t η ≡ t
         assoc : {σ : Γ –[ 𝓜 ]→ Δ} {ς : Δ –[ 𝓜 ]→ Θ} {t : 𝓜 α Γ} →
                   μ (μ t σ) ς ≡ μ t (λ v → μ (σ v) ς)
```

Simultaneous substitution μ and associativity assoc reduces to single-variable substitution and the substitution lemma, respectively

```
[_/] : 𝓜 α Γ → 𝓜 β (α · Γ) → 𝓜 β Γ
[ s /] t = μ t (λ new → s; (old v) → η v)
```

```
   [r/] ([s/] t)
≡ [ [r/] s /] ([r/] t)
```

The categorical viewpoint leads to an abstract, natural characterisation of substitution structure



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Initial pointed $\sum$-algebras encode syntax with variables

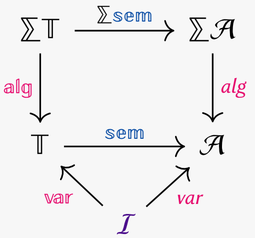
Signature endofunctor $\sum$ captures operator arities

For example, $\sum_{\text{Mon}}(\mathcal{A}) \triangleq \mathbb{1} + \mathcal{A} \times \mathcal{A}$

Algebras $\sum\mathcal{A} \to \mathcal{A}$ capture constructors of the syntax

For example, $[\text{unit}, \text{mult}] : (\mathbb{1} + \mathcal{A} \times \mathcal{A}) \to \mathcal{A} = \sum_{\text{Mon}}(\mathcal{A}) \to \mathcal{A}$

Initial pointed $\sum$-algebras $\sum\mathbb{T} \to \mathbb{T} \leftarrow \mathcal{I}$ capture structural recursion



$$\mathtt{sem}\,(\mathtt{alg}(t)) = \mathit{alg}\,(\textstyle\sum\mathtt{sem}(t))$$

$$\mathtt{sem}(\mathtt{var}\,(v)) = \mathit{var}\,(v)$$

Syntactic operations are constructed and proved correct using initiality

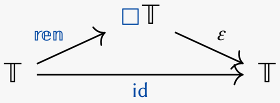
Syntactic operations can often be brought to the form $\mathbb{T} \to \mathcal{A}$

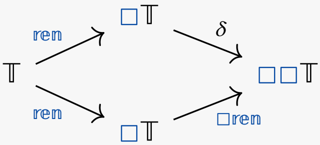
Suffices to show that $\mathcal{A}$ is a $(\Sigma + \mathcal{I})$-algebra

$$\mathsf{ren} : \mathbb{T} \to \Box\mathbb{T} \qquad \mathsf{sub} : \mathbb{T} \to \llparenthesis \mathbb{T}, \mathbb{T} \rrparenthesis$$

Correctness laws equate maps of the form $\mathbb{T} \to \mathcal{A}$

Suffices to show that edges are $(\Sigma + \mathcal{I})$-algebra homomorphisms

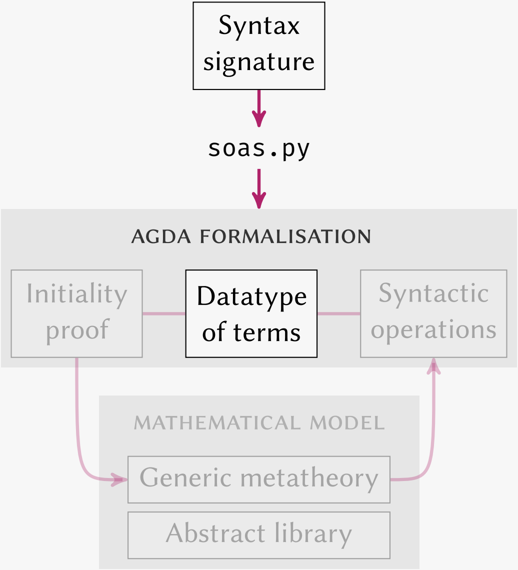




Coalgebra and monoid structure given by categorical reasoning

Compositional constructions and diagrammatic proofs

The substitution structure is defined and proved generically over any second-order signature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Everything starts with a syntax description file

```
syntax STLC | Λ

type
  N   : 0-ary
  _↣_ : 2-ary | r30

term
  app : (α ↣ β)  α  →  β     | _$_ l20
  lam : α.β         →  α ↣ β | ƛ_  r10
```

Type syntax

data ΛT : Set where
  N : ΛT
  _↣_ : $\Lambda T \to \Lambda T \to \Lambda T$

Term syntax

data $\Lambda : \Lambda T \to \mathsf{Ctx}\ \Lambda T \to$ Set where
  var : $\mathcal{I}\ \alpha\ \Gamma \to \Lambda\ \alpha\ \Gamma$
  _\$_ : $\Lambda\ (\alpha \succ \beta)\ \Gamma \to \Lambda\ \alpha\ \Gamma \to \Lambda\ \beta\ \Gamma$
  ƛ_ : $\Lambda\ \beta\ (\alpha \cdot \Gamma) \to \Lambda\ (\alpha \succ \beta)\ \Gamma$

Operator symbols

data Λ_o : Set where
  $\mathsf{app}_o : \{\alpha\ \beta : \Lambda T\} \to \Lambda_o$
  $\mathsf{lam}_o : \{\alpha\ \beta : \Lambda T\} \to \Lambda_o$

Signature

$\Lambda{:}\mathsf{Sig} : \Lambda_o \to \mathsf{List}\ (\mathsf{Ctx} \times \Lambda T) \times \Lambda T$
$\Lambda{:}\mathsf{Sig} = \lambda\ \{\ \mathsf{app}_o \to [([],\alpha \succ \beta),([],\alpha)],\ \beta$
$\quad ;\ \mathsf{lam}_o \to [([\alpha],\beta)],\ \alpha \succ \beta\ \}$

The signature determines the endofunctor $\mathbb{\Sigma}$ and its algebras

$$\mathbb{\Sigma} : \mathsf{Family}_s \to \mathsf{Family}_s$$
$$\mathbb{\Sigma}\ \mathcal{X}\ \alpha\ \Gamma = \Sigma[\ o \in \Lambda_o\ ]\ (\alpha \equiv \mathsf{Sort}\ o \times \mathsf{Arg}\ (\mathsf{Arity}\ o)\ \mathcal{X}\ \Gamma)$$

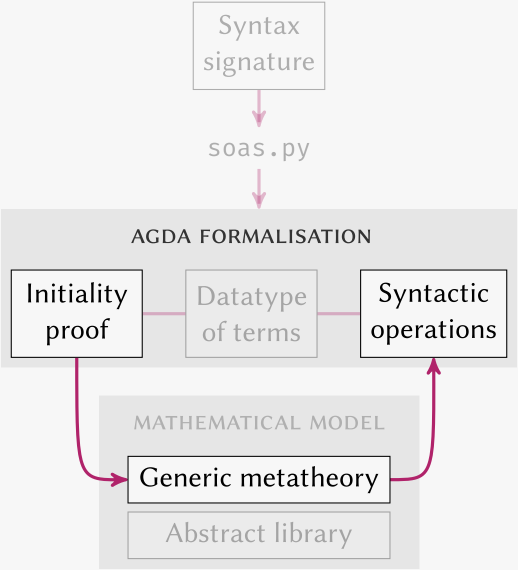
Associate an operator symbol with a tuple of $\mathcal{X}$-terms

The operator arity determines the type and context of subterms

$$f : \mathcal{X}\ (\alpha \rightarrowtail \beta)\ \Gamma,\ a : \mathcal{X}\ \alpha\ \Gamma \vdash \quad (\mathsf{app}_o, \mathsf{refl}, (f, a)) \quad : \mathbb{\Sigma}\ \mathcal{X}\ \beta\ \Gamma$$
$$b : \mathcal{X}\ \beta\ (\alpha \cdot \Gamma) \vdash \quad (\mathsf{lam}_o, \mathsf{refl}, b) \quad : \mathbb{\Sigma}\ \mathcal{X}\ (\alpha \rightarrowtail \beta)\ \Gamma$$

Pointed $\mathbb{\Sigma}$-algebras $\mathbb{\Sigma}\mathcal{X} \rightharpoonup \mathcal{X} \leftharpoondown \mathcal{I}$ carry the syntactic structure

The initial such algebra will also have substitution structure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The inductive type of terms is the initial pointed $\sum$-algebra

Translation of Λ to any algebra $(\mathcal{A}, \mathit{var} : \mathcal{I} \rightarrow \mathcal{A}, \mathit{alg} : \sum\mathcal{A} \rightarrow \mathcal{A})$
Captures all forms of recursive definitions on the syntax

```
sem : Λ → 𝒜
sem (var v)     = var v
sem (app g a) = alg (appₒ, refl, (sem g , sem a))
sem (lam b)     = alg (lamₒ, refl, sem b)
```

Uniqueness of sem among all homomorphisms $(g, \langle \mathit{var} \rangle, \langle \mathit{alg} \rangle)$
Captures all forms of inductive equality proofs on the syntax

```
sem! : (t : Λ α Γ) → sem t ≡ g t
sem! (var v)                                   = sym ⟨var⟩
sem! (app f a) rewrite sem! f | sem! a = sym ⟨alg⟩
sem! (lam b)     rewrite sem! b             = sym ⟨alg⟩
```

The initiality proof for the syntax instantiates the generic metatheory

```
open import SOAS.Metatheory Λ:Sig Λ 𝕤𝕖𝕞 𝕤𝕖𝕞!

⟦_⟧ : Λ α Γ ⟶ (⟦ Γ ⟧ᶜ → ⟦ α ⟧ᵗ)
⟦_⟧ = ΣAlg.𝕤𝕖𝕞 record { … }

sub-lemma : {b : Λ β (α ∙ Γ)}{a : Λ α Γ} (γ : ⟦ Γ ⟧ᶜ) →
            ⟦ [a/] b ⟧ γ ≡ ⟦ b ⟧ (γ ⁺ ⟦ a ⟧)

data _⇝_ : Λ α Γ ⟶ Λ α Γ ⟶ Set where
  β-ƛ : {b : Λ β (α ∙ Γ)}     {a : Λ α Γ} → (ƛ b) $ a ⇝ [ a /] b
  ζ-$ : {f g : Λ (α ↣ β) Γ} {a : Λ α Γ} → f          ⇝ g
                                         → f $ a      ⇝ g $ a

sound : {t s : Λ α Γ} → t ⇝ s → (γ : ⟦ Γ ⟧ᶜ) → ⟦ t ⟧ γ ≡ ⟦ s ⟧ γ
sound (ζ-$ r)       γ rewrite sound r γ = refl
sound (β-ƛ {t}{b}) γ rewrite sub-lemma t b
  = cong ⟦ b ⟧ (dext λ{ new → refl ; (old v) → refl })
```

Move from language design to
valuable metatheory in seconds!

Example: first-order logic

```
syntax F

type
  *   : 0-ary
  N   : 0-ary


term
  false         → * | ⊥
  and : *  *    → * | _∧_
  not : *       → * | ¬_
  all : N.*     → * | ∀̲_  ...


theory
  ‘and’ commutative, idempotent
  ‘false’ annihilates ‘and’
  (DM∧)  a  b ▷ not (and (a, b)) = or (not(a), not(b))
  (∧P∀ᴸ) p : *  q : N.* ▷ and (p, all(x.q[x]))
                        = all (x. and(p,q[x]))  ...
```

data FT : Set where
  ★ : FT
  N : FT

data F : FT → Ctx FT → Set where
  var : $\mathcal{I}\ \alpha\ \Gamma \to \Lambda\ \alpha\ \Gamma$
  ⊥ : F ★ Γ
  _∧_ : F ★ Γ → F ★ Γ → F ★ Γ
  ¬_ : F ★ Γ → F ★ Γ
  ∀̲_ : F ★ (N · Γ) → F ★ Γ ...

Example: first-order logic

```
data _▷_⊢_≈ₐ_ : ∀(𝔐 Γ) → F 𝔐 α Γ → F 𝔐 α Γ → Set where
  ∧C    : ⁅ * ⁆ ⁅ * ⁆      ▷ ∅ ⊢ 𝔞 ∧ 𝔟            ≈ₐ 𝔟 ∧ 𝔞
  ⊥X∧ᴸ  : ⁅ * ⁆            ▷ ∅ ⊢ ⊥ ∧ 𝔞            ≈ₐ ⊥
  DM∧   : ⁅ * ⁆ ⁅ * ⁆      ▷ ∅ ⊢ ¬ (𝔞 ∧ 𝔟)        ≈ₐ (¬ 𝔞) ∨ (¬ 𝔟)
  ∧P∀ᴸ  : ⁅ * ⁆ ⁅ N ⊩ * ⁆  ▷ ∅ ⊢ 𝔞 ∧ (∀̲ 𝔟⟨ x₀ ⟩) ≈ₐ ∀̲ (𝔞 ∧ 𝔟⟨ x₀ ⟩)    ...

ax_with_   : {s t : F 𝔐 α Π}          → 𝔐 ▷ Π ⊢ s ≈ₐ t →
             (ζ : MSub Γ 𝔐 𝔑) → 𝔑 ▷ Π ⊢ msub s ζ ≈ msub t ζ

cong[_]in_ : {s t : F 𝔐 β (Π ∔ Γ)}       → 𝔐 ▷ (Π ∔ Γ) ⊢ s ≈ₐ t →
             (u : F (𝔐 ⁅ Π ⊩ β ⁆) α Γ) → 𝔐 ▷ Γ ⊢ msub₁ u s ≈ msub₁ u t


∧P∀ᴿ : ⁅ N ⊩ * ⁆ ⁅ * ⁆ ⊢ (∀̲ 𝔞⟨ x₀ ⟩) ∧ 𝔟 ≈ ∀̲ (𝔞⟨ x₀ ⟩ ∧ 𝔟)
∧P∀ᴿ = begin
    (∀̲ 𝔞⟨ x₀ ⟩) ∧ 𝔟    ≈⟨ ax ∧C with ⟪ (∀̲ 𝔞⟨ x₀ ⟩) ◂ 𝔟 ⟫ ⟩
    𝔟 ∧ (∀̲ 𝔞⟨ x₀ ⟩)    ≈⟨ ax ∧P∀ᴸ with ⟪ 𝔟 ◂ 𝔞⟨ x₀ ⟩ ⟫ ⟩
    ∀̲ (𝔟 ∧ 𝔞⟨ x₀ ⟩)    ≈⟨ cong[ ax ∧C with ⟪ 𝔟 ◂ 𝔞⟨ x₀ ⟩ ⟫ ]in ∀̲ (◌ᶜ⟨ x₀ ⟩) ⟩
    ∀̲ (𝔞⟨ x₀ ⟩ ∧ 𝔟)    ■
```

Conclusions

A signature captures the full syntactic structure of a language
No definition or proof requires particular, syntax-specific insight

Categorical viewpoint lends generality and deep understanding
Capture and abstract over common constructions from literature

Second-order extensions follow naturally from the theory
Further analysis of metasubstitution is ongoing work

Generality, practicality, efficiency – choose three!

Source code, documentation, and examples
can be found on the project page:

`https://tinyurl.com/agda-soas`

Give it a try!